

RELATIVE HOMOLOGICAL ALGEBRA, WALDHAUSEN K -THEORY, AND QUASI-FROBENIUS CONDITIONS.

ANDREW SALCH

ABSTRACT. We produce the structure of a Waldhausen category on any (relative) abelian category in which the contractible objects are the (relatively) projective objects, under appropriate assumptions. The associated K -theory groups are “stable algebraic G -theory,” which in degree zero form a certain stable representation group, and in higher degrees measure more subtle invariants of stable representation theory in that (relative) abelian category. We prove both some existence and nonexistence results about such Waldhausen category structures, including the fact that, while it was known that the category of R -modules admits a model category structure if R is quasi-Frobenius, that assumption is required even to get a Waldhausen category structure with cylinder functor—i.e., Waldhausen categories do not offer a more general framework than model categories for studying stable representation theory of rings. Finally, we produce some localization results, resulting in long exact sequences and spectral sequences for computing stable algebraic G -theory of finite-dimensional Hopf algebras.

CONTENTS

1. Introduction.	1
2. Waldhausen category structures from allowable classes on abelian categories.	4
2.1. Definitions.	4
2.2. The Waldhausen category structure on an abelian category associated to a pair of allowable classes.	10
2.3. Pushing forward and pulling back allowable classes.	13
3. Absolute and relative quasi-Frobenius conditions.	16
3.1. Definitions.	16
3.2. Existence of cylinder functors.	17
3.3. Compatibility of cylinder functors, and using the Fibration Theorem.	18
4. Applications.	23
4.1. The relationship between stable G -theory and other G -theories and K -theories.	23
4.2. Applications to algebras.	27
References	31

1. INTRODUCTION.

Waldhausen’s paper [14] defines several kinds of categorical structure which are meaningful for algebraic K -theory. A *category with cofibrations and weak equivalences*, also called a *Waldhausen category*, has just enough structure for Waldhausen’s machinery to

produce an associated K -theory infinite loop space. A Waldhausen category which satisfies additional axioms and/or has additional structure will have better properties which e.g. make the problem of actually computing the associated K -theory more tractable. For example, a Waldhausen category satisfying the “extension axiom” and the “saturation axiom” and equipped with an additional structure called a “cylinder functor” admits Waldhausen’s *Localization Theorem* (see [14]), a computationally powerful result that describes the sense in which localizations of the Waldhausen category induce long exact sequences in the K -theory groups.

Meanwhile, in stable representation theory, one regards the projective modules over a ring as “contractible,” and maps of modules that factor through projective modules are regarded as “nulhomotopic.” This suggests that the category of modules over a ring perhaps has a Waldhausen category structure in which the objects weakly equivalent to zero—the contractible objects—are precisely the projective modules. More generally, one has the tools of *relative homological algebra*: if one chooses a sufficiently well-behaved class of objects in an abelian category, one can do a form of homological algebra in which the chosen class of objects plays the role of projective objects. Given an abelian category and a class of relative projective objects, one wants to know if there is a natural Waldhausen category structure on that abelian category, such that the contractible objects are precisely the relative projectives. One also wants to know how many extra axioms are satisfied by, and how much additional structure is admitted by, such a Waldhausen category.

In this paper we prove three theorems that answer the above questions, and explain fundamental properties of the relationships between Waldhausen K -theory, relative homological algebra, and stable representation theory:

- (1) Def.-Prop. 2.2.5: Given an abelian category \mathcal{C} and a sufficiently nice pair of allowable classes E, F in \mathcal{C} , there exists a Waldhausen category structure on \mathcal{C} whose weak equivalences are the E -stable equivalences and whose cofibrations are the F -monomorphisms. In particular, the E -projective objects are precisely the contractible objects in this Waldhausen category. This Waldhausen category satisfies the saturation axiom and the extension axiom.
- (2) Thm. 3.2.1: If \mathcal{C} has enough injectives, then \mathcal{C} has a cylinder functor satisfying the cylinder axiom if and only if \mathcal{C} obeys a certain generalized quasi-Frobenius condition: every object must functorially embed in an E -projective object by an F -monomorphism.
- (3) Thm. 3.3.5: If $E = F$, \mathcal{D} is a sufficiently nice reflective abelian subcategory of \mathcal{C} , both \mathcal{C} and \mathcal{D} obey a generalized quasi-Frobenius condition, and the localization functor from \mathcal{C} to \mathcal{D} is *relatively* quasi-Frobenius in an appropriate sense, then Waldhausen’s Fibration Theorem applies, giving a long exact sequence relating the K -groups of \mathcal{C} and \mathcal{D} and the Waldhausen category of objects in \mathcal{C} whose images in \mathcal{D} are contractible.

“Quasi-Frobenius conditions” appear prominently throughout this paper. Recall that a ring R is said to be quasi-Frobenius if every projective R -module is injective and vice versa. The appearance of these conditions in connection with Waldhausen K -theory stems from the theorem of Faith and Walker (see [3] for a good account of this and related theorems):

Theorem 1.0.1. (Faith-Walker.) *A ring R is quasi-Frobenius if and only if every R -module embeds in a projective R -module.*

If a category of R -modules admits a Waldhausen category structure in which the projective R -modules are the contractible objects, and that Waldhausen category has a cylinder

functor satisfying the cylinder axiom, then in particular, when one applies that cylinder functor to the projection $M \rightarrow 0$ for an R -module M , one gets an embedding (by a cofibration) of M into a projective R -module. So the kind of structures we are looking for, cylinder functors on Waldhausen categories of modules in which projectives (or relative projectives) are the contractible objects, can only exist in the presence of a quasi-Frobenius condition. The content of our Thm. 3.2.1 is that the most obvious (from our point of view) “relative” analogue of the quasi-Frobenius condition in fact is necessary and sufficient in order to get a cylinder functor on these Waldhausen categories that arise from relative homological algebra.

Here is one point of view on the significance of Thm. 3.2.1. It has been known for a long time, e.g. as described in [4], that when R is a quasi-Frobenius ring, there exists a model category structure on the category of R -modules in which the cofibrations are the injections and the weak equivalences are the stable equivalences of modules. Constructing this model category structure uses the quasi-Frobenius condition in an essential way. But a Waldhausen category structure on R -modules is weaker, less highly-structured, than a model category structure; so one might hope that, even in the absence of the quasi-Frobenius condition on R , one could put the structure of a Waldhausen category on R -modules, such that the cofibrations are the injections and the weak equivalences are the stable equivalences of modules. As a consequence of Thm. 3.2.1, one only gets a Waldhausen category structure with cylinder functor on the category of R -modules if R is quasi-Frobenius. So *the category of R -modules admits a model category structure as desired if and only if it admits a Waldhausen category structure with cylinder functor as desired.*

Our Thm. 3.3.5 describes the necessary generalized quasi-Frobenius properties in order for localizing one of our Waldhausen categories to yield a fibration as in Waldhausen’s Fibration Theorem, and hence a computable long exact sequence in K -groups. Here is one point of view on what this theorem accomplishes: in [10], Quillen proves that localization of an abelian category at a Serre subcategory (i.e., a subcategory closed under extensions, subobjects, and quotient objects) induces a long exact sequence in algebraic K -groups. Most interesting subcategories of abelian categories are not Serre, however; for example, if $R \rightarrow S$ is a surjection of rings, the category of S -modules is (by restriction of scalars) a reflective, but typically non-Serre, subcategory of the category of R -modules. Thm. 3.3.5 tells us, however, that if we work with the Waldhausen K -theory described above (which is not at all the same as ordinary algebraic K -theory!), then under conditions identified in the theorem, localization at a subcategory can induce a long exact sequence in K -groups even when that subcategory fails to be Serre.

Finally, we describe some practical and computational consequences of all of this:

- In Cor. 3.3.8, we construct a spectral sequence for computing Waldhausen K -theory of an abelian category (using the Waldhausen category structure described above) equipped with a sufficiently nice filtration by subcategories satisfying appropriate quasi-Frobenius conditions.
- In Thm. 4.2.4, we prove that certain extensions of Hopf algebras over a finite field induce long exact sequences in “stable algebraic G -theory.” Stable algebraic G -theory is defined in Def. 3.3.6. It is the name we give to the Waldhausen K -theory associated to the Waldhausen category of modules over a ring (or scheme, stack, ...) in which the cofibrations are the monomorphisms and the weak equivalences are the stable equivalences.

- If a conjecture about twisted deformation theory which we describe in Remark 4.2.6 can be proven, then Thm. 4.2.4 in fact holds for *all* extensions of graded connective co-commutative finite-dimensional Hopf algebras over a finite field. One then has the result that *every extension of finite-dimensional co-commutative connective Hopf algebras over a finite field induces a long exact sequence in stable algebraic G -groups*.
- As a consequence of Thm. 4.2.4, one has Cor. 4.2.7, which constructs a spectral sequence for computing the stable algebraic G -theory of a (sufficiently nice) filtered Hopf algebra from the stable algebraic G -theory of the filtration quotient Hopf algebras.

The long exact sequences and spectral sequence arising from Thm. 4.2.4 and Cor. 4.2.7 have the consequence that, if one can compute, for example, the stable algebraic G -theory of the exterior algebra $\mathbb{F}_2[x]/x^2$, then one can compute the stable algebraic G -theory of every connective co-commutative Hopf algebra over \mathbb{F}_2 admitting a Hopf algebra composition series whose layers are all isomorphic to $\mathbb{F}_2[x]/x^2$ —for example, every finite-dimensional exterior algebra over \mathbb{F}_2 , and every subalgebra $A(n)$ of the mod 2 Steenrod algebra. Making this first nontrivial computation of $(G_{st})_*(\mathbb{F}_2[x]/x^2)$, however, is quite difficult. We have an approach which makes use of a detailed study of the effect on K -groups of changing the class of cofibrations of a Waldhausen category, but it involves difficult computations in a relative Postnikov tower and only yields low-dimensional information; and, aside from a comment in Remark 4.1.4, it is beyond the scope of this present paper.

We note that Thm. 4.2.4 uses the main result of our preprint [12] in an essential way, and Prop.-Def. 2.2.5 uses a result from our preprint [11] in an essential way.

We would not have written this paper or thought about any of these issues if not for conversations we had with Crichton Ogle, who taught us a great deal about Waldhausen K -theory during the summer of 2012. We are grateful to C. Ogle for his generosity in teaching us about this subject.

2. WALDHAUSEN CATEGORY STRUCTURES FROM ALLOWABLE CLASSES ON ABELIAN CATEGORIES.

2.1. Definitions. This subsection, mostly consisting of definitions, is entirely review and there are no new results or definitions in it, with the exception of Def. 2.1.8 and Def.-Prop. 2.1.10.

Throughout this subsection, let \mathcal{C} be an abelian category.

We begin with the definition of an allowable class. An allowable class is the structure one needs to specify on \mathcal{C} in order to have a notion of relative homological algebra in \mathcal{C} .

Definition 2.1.1. *An allowable class in \mathcal{C} consists of a collection E of short exact sequences in \mathcal{C} which is closed under isomorphism of short exact sequences and which contains every short exact sequence in which at least one object is the zero object of \mathcal{C} . (See section IX.4 of [6] for this definition and basic properties.)*

The usual “absolute” homological algebra in an abelian category \mathcal{C} is recovered by letting the allowable class E consist of *all* short exact sequences in \mathcal{C} .

Once one chooses an allowable class E , one has the notion of monomorphisms relative to E , or “ E -monomorphisms,” and epimorphisms relative to E , or “ E -epimorphisms.”

Definition 2.1.2. *Let E be an allowable class in \mathcal{C} . A monomorphism $f : M \rightarrow N$ in \mathcal{C} is called an E -monomorphism or an E -monic if the short exact sequence*

$$0 \rightarrow M \xrightarrow{f} N \rightarrow \operatorname{coker} f \rightarrow 0$$

is in E .

Dually, an epimorphism $g : M \rightarrow N$ is called an E -epimorphism or an E -epic if the short exact sequence

$$0 \rightarrow \ker f \rightarrow M \xrightarrow{f} N \rightarrow 0$$

is in E .

In the absolute case, the case that E is all short exact sequences in C , the E -monomorphisms are simply the monomorphisms, and the E -epimorphisms are simply the epimorphisms.

Projective and injective objects are at the heart of homological algebra. In relative homological algebra, one has the notion of relative projectives, or E -projectives: these are simply the objects which lift over every E -epimorphism. The E -injectives are defined dually.

Definition 2.1.3. Let E be an allowable class in C . An object X of C is said to be an E -projective if, for every diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

in which f is an E -epic, there exists a morphism $X \rightarrow M$ making the above diagram commute.

Dually, an object X of C is said to be an E -injective if, for every diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \\ & & X \end{array}$$

in which f is an E -monic, there exists a morphism $N \rightarrow X$ making the above diagram commute.

When the allowable class E is clear from context we sometimes refer to E -projectives and E -injectives as relative projectives and relative injectives, respectively.

In the absolute case, the case that E is all short exact sequences in C , the E -projectives are simply the projectives, and the E -injectives are simply the injectives.

Once one has a notion of relative projectives, one has a reasonable notion of a stable equivalence or, loosely, a “homotopy” between maps, as studied (usually in the absolute case, where E -projectives are simply projectives) in stable representation theory:

Definition 2.1.4. Let E be an allowable class in C . Let $f, g : M \rightarrow N$ be morphisms in C . We say that f and g are E -stably equivalent and we write $f \simeq g$ if $f - g$ factors through an E -projective object of C .

One then has the notion of stable equivalence of objects, or loosely, “homotopy equivalence”:

Definition 2.1.5. We say that a map $f : M \rightarrow N$ is a E -stable equivalence if there exists a map $h : N \rightarrow M$ such that $f \circ h \simeq \text{id}_N$ and $h \circ f \simeq \text{id}_M$.

In the absolute case where E consists of all short exact sequences in C , this is the usual notion of stable equivalence of modules over a ring. Over a Hopf algebra over a field, stably equivalent modules have isomorphic cohomology in positive degrees, so if one is serious

about computing the cohomology of all finitely-generated modules over a particular Hopf algebra—such as the Steenrod algebra or the group ring of a Morava stabilizer group—it is natural to first compute the representation ring modulo stable equivalence. See [8] for this useful perspective (which motivates much of the work in this paper).

We now define the relative-homological-algebraic generalizations of an abelian categories having enough projectives or enough injectives. We provide an extra twist on this definitions as well, which we will need for certain theorems: sometimes we will need to know that, for example, not only does every object embed in an injective, but that we can choose such embeddings in a functorial way.

Definition 2.1.6. *Let E be an allowable class in \mathcal{C} . We say that \mathcal{C} has enough E -projectives if, for any object M of \mathcal{C} , there exists an E -epic $N \rightarrow M$ with N an E -projective. We say that \mathcal{C} has functorially enough E -projectives if \mathcal{C} has enough E -projectives and the choice of E -epimorphisms from E -projectives to each object of \mathcal{C} can be made functorially, i.e., there exists a functor $P : \mathcal{C} \rightarrow \mathcal{C}$ together with a natural transformation $\epsilon : P \rightarrow \text{id}_{\mathcal{C}}$ such that $P(X)$ is E -projective and $\epsilon(X) : P(X) \rightarrow X$ is an E -epimorphism for all objects X of \mathcal{C} , and such that, if $f : X \rightarrow Y$ is an E -epimorphism, then so is $P(f) : P(X) \rightarrow P(Y)$.*

Dually, we say that \mathcal{C} has enough E -injectives if, for any object M of \mathcal{C} , there exists an E -monic $M \rightarrow N$ with N an E -injective. We say that \mathcal{C} has functorially enough E -injectives if \mathcal{C} has enough E -injectives and the choice of E -monomorphisms from E -injectives to each object of \mathcal{C} can be made functorially, i.e., there exists a functor $I : \mathcal{C} \rightarrow \mathcal{C}$ together with a natural transformation $\eta : \text{id}_{\mathcal{C}} \rightarrow I$ such that $I(X)$ is E -injective and $\eta(X) : X \rightarrow I(X)$ is an E -monomorphism for all objects X of \mathcal{C} , and such that, if $f : X \rightarrow Y$ is an E -monomorphism, then so is $I(f) : I(X) \rightarrow I(Y)$.

Our need to have abelian categories with *functorially* enough injectives or projectives is only due to Waldhausen’s definitions of cylinder functors and resulting theorems, in [14], demanding that cylinder functors actually be *functors*. It seems likely that one can do away with this assumption and still prove analogues of Waldhausen’s results that use cylinders (e.g. the Fibration Theorem) by mimicking the situation in model category theory: there, one knows that any morphism has a factorization into a cofibration followed by an acyclic fibration, but such factorizations are not provided in a functorial way. We don’t pursue that angle in this paper, however.

Finally, we have our first definition of a quasi-Frobenius condition:

Definition 2.1.7. *Let E be an allowable class in \mathcal{C} . We will call E a quasi-Frobenius allowable class if the E -projectives are exactly the E -injectives. If the allowable class consisting of all short exact sequences in \mathcal{C} is a quasi-Frobenius class, then we will simply say that \mathcal{C} is quasi-Frobenius.*

Here are some important examples of allowable classes in abelian categories:

- As described above, the usual “absolute” homological algebra in an abelian category \mathcal{C} is recovered by letting the allowable class E consist of *all* short exact sequences in \mathcal{C} ; then the E -projectives are the usual projectives, etc. Note that, if E is an arbitrary allowable class in \mathcal{C} , then any projective object is an E -projective object, but the converse is not necessarily true.
- There is another “trivial” case of an allowable class: if we let E consist of only the short exact sequences in \mathcal{C} in which at least one of the objects is the zero object, then the E -epics consists of all identity maps as well as all projections to the zero object, and the E -monics consist of all identity maps as well as all inclusions of the

zero object. As a consequence all objects are both E -injectives and E -projectives, and E is a quasi-Frobenius allowable class.

- Suppose \mathcal{C}, \mathcal{D} are abelian categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ is an additive functor. Then we can let E be the allowable class in \mathcal{C} consisting of the short exact sequences which are sent by F to split short exact sequences in \mathcal{D} . If F has a left (resp. right) adjoint G then objects of \mathcal{C} of the form GFX (resp. FGX) are E -projectives (resp. E -injectives) and the counit map $GFX \rightarrow X$ of the comonad GF (resp. the unit map $X \rightarrow GFX$ of the monad GF) is an E -epic (resp. E -monic), hence \mathcal{C} has enough E -projectives (resp. enough E -injectives). These ideas are in [6].

For example, if R is a ring and \mathcal{C} the category of R -modules and \mathcal{D} the category of abelian groups, and F the forgetful functor, then E is the class of short exact sequences of R -modules whose underlying short exact sequences of abelian groups are split. The R -modules of the form $R \otimes_{\mathbb{Z}} M$, for M an R -module, are E -projectives.

Here is a new definition which makes many arguments involving allowable classes substantially smoother:

Definition 2.1.8. *An allowable class E is said to have retractile monics if, whenever $g \circ f$ is an E -monic, f is also an E -monic.*

Dually, an allowable class E is said to have sectile epics if, whenever $g \circ f$ is an E -epic, g is also an E -epic.

The utility of the notion of “having sectile epics” comes from the following fundamental theorem of relative homological algebra, due to Heller (see [6]), whose statement is slightly cleaner is one is willing to use the phrase “having sectile epics.” The consequence of Heller’s theorem is that, in order to specify a “reasonable” allowable class in an abelian category, it suffices to specify the relative projective objects associated to it.

Theorem 2.1.9. *If \mathcal{C} is an abelian category and E is an allowable class in \mathcal{C} with sectile epics and enough E -projectives, then an epimorphism $M \rightarrow N$ in \mathcal{C} is an E -epic if and only if the induced map $\text{hom}_{\mathcal{C}}(P, M) \rightarrow \text{hom}_{\mathcal{C}}(P, N)$ of abelian groups is an epimorphism for all E -projectives P .*

Heller’s theorem suggests the following construction, which as far as we know, is new (but unsurprising): if E is an allowable class, we can construct a “sectile closure of E ” which has the same relative projectives and the same stable equivalences but which has sectile epics. Here are the specific properties of this construction (we have neglected to write out proofs of these properties because the proofs are so elementary):

Definition-Proposition 2.1.10. *Let \mathcal{C} be an abelian category, E an allowable class in \mathcal{C} . Let E_{sc} be the allowable class in \mathcal{C} consisting of the exact sequences*

$$X \rightarrow Y \rightarrow Y/X$$

such that the induced map

$$\text{hom}_{\mathcal{C}}(P, Y) \rightarrow \text{hom}_{\mathcal{C}}(P, Y/X)$$

is a surjection of abelian groups for every E -projective P . We call E_{sc} the sectile closure of E . The allowable class E_{sc} has the following properties:

- E_{sc} has sectile epics.
- An object of \mathcal{C} is an E -projective if and only if it is an E_{sc} -projective.
- If f, g are two morphisms in \mathcal{C} then f and g are E -stably equivalent if and only if they are E_{sc} -stably equivalent.

- If X, Y are two objects in \mathcal{C} then X and Y are E -stably equivalent if and only if they are E_{sc} -stably equivalent.
- $(E_{sc})_{sc} = E_{sc}$.
- If E, F are two allowable classes in \mathcal{C} and $F \subseteq E$ then $F_{sc} \subseteq E_{sc}$.

Of course there is a construction dual to the sectile closure, a *retractile closure*, with dual properties, but with a less straightforward relationship to stable equivalence, since stable equivalence is defined in terms of projectives, not injectives.

We now recall Waldhausen's definitions:

Definition 2.1.11. A pointed category \mathcal{C} with finite pushouts equipped with a specified class of cofibrations and a specified class of weak equivalences, both closed under composition, is called a Waldhausen category if the following axioms are satisfied:

- **(Cof 1.)** The isomorphisms in \mathcal{C} are cofibrations.
- **(Cof 2.)** For every object X of \mathcal{C} , the map $\text{pt.} \rightarrow \mathcal{C}$ is a cofibration. (We write pt. for the zero object of \mathcal{C} .)
- **(Cof 3.)** If $X \rightarrow Y$ is a morphism in \mathcal{C} and $X \rightarrow Z$ is a cofibration, then the canonical map $Y \rightarrow Y \amalg_X Z$ is a cofibration.
- **(Weq 1.)** The isomorphisms in \mathcal{C} are weak equivalences.
- **(Weq 2.)** If

$$(2.1) \quad \begin{array}{ccccc} Y & \longleftarrow & X & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longleftarrow & X' & \longrightarrow & Z' \end{array}$$

is a commutative diagram in \mathcal{C} in which the maps $X \rightarrow Y$ and $X' \rightarrow Y'$ are cofibrations and all three vertical maps are weak equivalences, then the induced map $Y \amalg_X Z \rightarrow Y' \amalg_{X'} Z'$ is a weak equivalence.

Ultimately, if \mathcal{C} is a Waldhausen category, then what one typically wants to understand is $|wS.\mathcal{C}|$, the geometric realization of the simplicial category $wS.\mathcal{C}$ constructed by Waldhausen in [14]. The K -groups of \mathcal{C} are defined as the homotopy groups of the loop space $\Omega |wS.\mathcal{C}|$:

$$\begin{aligned} \pi_{n+1}(|wS.\mathcal{C}|) &\cong \pi_n(\Omega |wS.\mathcal{C}|) \\ &\cong K_n(\mathcal{C}). \end{aligned}$$

If \mathcal{C} is a Waldhausen category then the following axioms may or may not be satisfied:

- Definition 2.1.12.**
- **(Saturation axiom.)** If f, g are composable maps in \mathcal{C} and two of $f, g, g \circ f$ are weak equivalences then so is the third.
 - **(Extension axiom.)** If

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Y/X \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Y'/X' \end{array}$$

is a map of cofiber sequences and the maps $X \rightarrow X'$ and $Y/X \rightarrow Y'/X'$ are weak equivalences then so is the map $Y \rightarrow Y'$.

Definition 2.1.13. If C is a Waldhausen category, a cylinder functor on C is a functor from the category of arrows $f : X \rightarrow Y$ in C to the category of diagrams of the form

$$\begin{array}{ccccc} X & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & Y \\ & \searrow f & \downarrow p & \swarrow \text{id} & \\ & & Y & & \end{array}$$

in C satisfying the two conditions:

- **(Cyl 1.)** If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a commutative diagram in C in which the vertical maps are cofibrations (resp. weak equivalences), then in the canonically associated commutative diagram

$$\begin{array}{ccc} X' \amalg Y' & \xrightarrow{j'_1 \amalg j'_2} & T(f') \\ \downarrow & & \downarrow \\ X \amalg Y & \xrightarrow{j_1 \amalg j_2} & T(f) \end{array}$$

the vertical maps are cofibrations (resp. weak equivalences).

- **(Cyl 2.)** $T(\text{pt.} \rightarrow Y) = Y$ and the maps p and j_2 are the identity map on Y .

A Waldhausen category with cylinder functor may or may not satisfy the additional condition:

- **(Cylinder axiom.)** For any map f in C , the map p is a weak equivalence.

The idea here is that a cylinder functor satisfying the cylinder axiom acts very much like the mapping cylinder construction from classical homotopy theory—or, more generally, like fibrant replacement in a model category. Some of Waldhausen’s most powerful results in [14] have proofs of a sufficiently homotopy-theoretic flavor that they require that every Waldhausen category in sight has a cylinder functor obeying the cylinder axioms. A good example of this is Waldhausen’s Fibration Theorem, which we now recall:

Theorem 2.1.14. Fibration Theorem (Waldhausen). Suppose C, C_0 are Waldhausen categories with the same underlying category and the same underlying class of cofibrations. Suppose all of the following conditions are satisfied:

- Every weak equivalence in C is also a weak equivalence in C_0 .
- The weak equivalences in C_0 satisfy the cylinder, saturation, and extension axioms.
- C admits a cylinder functor satisfying the cylinder axiom and the same cylinder functor is also a cylinder functor on C_0 .

Then

$$|wS.X| \rightarrow |wS.C| \rightarrow |wS.C_0|$$

is a homotopy fibre sequence, where X is the full sub-Waldhausen-category of C generated by the objects that are weakly equivalent to pt. in C_0 . As a consequence, after looping and taking homotopy groups, we get the long exact sequence of K -groups:

$$\cdots \rightarrow K_{n+1}(C_0) \rightarrow K_n(X) \rightarrow K_n(C) \rightarrow K_n(C_0) \rightarrow K_{n-1}(X) \rightarrow \cdots$$

The question of when our Waldhausen categories given by allowable classes on abelian categories satisfy the required conditions for the Fibration Theorem to hold is the subject of most of this paper.

2.2. The Waldhausen category structure on an abelian category associated to a pair of allowable classes. In this subsection, we will prove that an abelian category equipped with a pair of allowable classes E, F admits a Waldhausen category structure in which the cofibrations are the F -monomorphisms and the weak equivalences are the E -stable equivalences. In this subsection we will also find conditions under which this Waldhausen category satisfies the extension and saturation axioms (see Def. 2.1.12 for definitions of these axioms). To get anywhere, we will need some lemmas:

Lemma 2.2.1. *A pullback of a surjective map of abelian groups is surjective.*

Proof. The forgetful functor from abelian groups to sets is a right adjoint, hence preserves limits. It also clearly preserves surjections. So the lemma is true if a pullback of a surjective maps of sets is surjective, which is an elementary exercise. \square

Lemma 2.2.2. *Let \mathcal{C} be an abelian category and let E be an allowable class with retractile monics. Then E -monics are closed under pushout in \mathcal{C} . That is, if $X \rightarrow Z$ is an E -monic and $X \rightarrow Y$ is any morphism in \mathcal{C} , then the canonical map $Y \rightarrow Y \amalg_X Z$ is an E -monic.*

Proof. Suppose $f : X \rightarrow Z$ is an E -monic and $X \rightarrow Y$ any morphism. We have the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Z & \longrightarrow & \operatorname{coker} f \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Y & \longrightarrow & Y \amalg_X Z & \longrightarrow & \operatorname{coker} f \longrightarrow 0 \end{array}$$

and hence, for every E -injective I , the induced commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(\operatorname{coker} f, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(Z, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(X, I) \longrightarrow 0 \\ \downarrow & & \cong \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(\operatorname{coker} f, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(Y \amalg_X Z, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(Y, I). \end{array}$$

Exactness of the top row follows from f being an E -monic together with E having retractile monics, hence E is its own retractile closure, hence E -monics are precisely the maps which induce a surjection after applying $\operatorname{hom}_{\mathcal{C}}(-, I)$ for every E -injective I . Now in particular we have a commutative square in the above commutative diagram:

$$\begin{array}{ccc} \operatorname{hom}_{\mathcal{C}}(Z, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(X, I) \\ \uparrow & & \uparrow \\ \operatorname{hom}_{\mathcal{C}}(Y \amalg_X Z, I) & \longrightarrow & \operatorname{hom}_{\mathcal{C}}(Y, I), \end{array}$$

which is a pullback square of abelian groups, by the universal property of the pushout. The top map in the square is a surjection, hence so is the bottom map, Lemma 2.2.1. So $\operatorname{hom}_{\mathcal{C}}(Y \amalg_X Z, I) \rightarrow \operatorname{hom}_{\mathcal{C}}(Y, I)$ is a surjection for every E -injective I . Again since E is its own retractile closure, this implies that $Y \rightarrow Y \amalg_X Z$ is an E -monic. \square

Lemma 2.2.3. *Suppose \mathcal{C} is an abelian category, E an allowable class in \mathcal{C} with retractile monics. Suppose \mathcal{C} has enough E -injectives. A composite of E -monomorphisms is an E -monomorphism.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be E -monomorphisms. Let I be an E -injective object equipped with a map $X \rightarrow I$. Then, since f is an E -monomorphism, $X \rightarrow I$ extends through f to a map $Y \rightarrow I$, which in turn extends through g since g is an E -monomorphism. So every map to an E -injective from Z extends through $g \circ f$. Now, by the dual of Heller's theorem 2.1.9, $g \circ f$ is an E -monomorphism. \square

Lemma 2.2.4. *Let \mathcal{C} be an abelian category and let E be an allowable class in \mathcal{C} . Then a composite of two E -stable equivalences in \mathcal{C} is an E -stable equivalence.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of E -stable equivalences. Then there exist E -projective objects P_X, P_Y of \mathcal{C} and morphisms

$$\begin{aligned} Y &\xrightarrow{f'} X, \\ Z &\xrightarrow{g'} Y, \\ X &\xrightarrow{i_X} P_X, \\ Y &\xrightarrow{i_Y} P_Y, \\ P_X &\xrightarrow{s_X} X, \text{ and} \\ P_Y &\xrightarrow{s_Y} Y \end{aligned}$$

such that

$$\begin{aligned} f' \circ f - \text{id}_X &= s_X \circ i_X \text{ and} \\ g' \circ g - \text{id}_Y &= s_Y \circ i_Y. \end{aligned}$$

Then we have

$$f' \circ g' \circ g \circ f - \text{id}_X = s_X \circ i_X + f' \circ s_Y \circ i_Y \circ f$$

so $f' \circ g' \circ g \circ f - \text{id}_X$ factors through the E -projective $P_X \oplus P_Y$. A similar argument applies to showing that $g \circ f \circ f' \circ g' - \text{id}_Z$ factors through an E -projective. So $g \circ f$ is an E -stable equivalence. \square

Definition-Proposition 2.2.5. *Let \mathcal{C} be an abelian category, let E, F be allowable classes in \mathcal{C} with $F \subseteq E$. Suppose each of the following conditions are satisfied:*

- F has retractile monics.
- E has retractile monics and sectile epics.
- \mathcal{C} has enough F -injectives.
- \mathcal{C} has enough E -projectives and enough E -injectives.
- Every E -projective object is E -injective.

Then there exists a Waldhausen category structure on \mathcal{C} in which the cofibrations are the F -monomorphisms and the weak equivalences are the E -stable equivalences. We write $\mathcal{C}_{E\text{-we}}^{F\text{-cof}}$ for this Waldhausen category. This Waldhausen category satisfies the saturation axiom and the extension axiom.

Proof. We check Waldhausen's axioms from Def. 2.1.11. In the case of an abelian category \mathcal{C} and allowable classes E, F with the stated classes of cofibrations and weak equivalences, axioms (Cof 1) and (Cof 2) and (Weq 1) are immediate. That the class of cofibrations is

closed under composition is Lemma 2.2.3. That the class of weak equivalences is closed under composition is Lemma 2.2.4. We show that the remaining axioms are satisfied:

- Axiom (Cof 3) is a consequence of Lemma 2.2.2.
- Axiom (Weq 2) is actually fairly substantial and takes some work to prove—enough so that we moved this work into a paper of its own, [11]. In Cor. 4.4 of that paper, we prove that, if $E = F$, C has enough E -projectives and enough E -injectives, E has sectile epics and retractile monics, and every E -projective object is E -injective, then C satisfies axiom (Weq 2). We refer the reader to that paper for the proof, which requires some work and a number of preliminary results, and would make the present paper much longer if we included it here.

Once we have the result for $F = E$, it follows for $F \subseteq E$, since if diagram 2.1 has its horizontal maps F -monomorphisms, the horizontal maps are also then E -monomorphisms.

- The saturation axiom follows easily from Lemma 2.2.4 together with the observation that, if $X \xrightarrow{f} Y$ is a E -stable equivalence and $Y \xrightarrow{f'} X$ is a morphism such that $f \circ f' - \text{id}_Y$ and $f' \circ f - \text{id}_X$ both factor through E -projective objects, then f' is also a E -stable equivalence. That is, E -stable equivalences have “up-to-equivalence inverses.”
- Finally, we handle the extension axiom. We begin by assuming that $E = F$, and that

$$(2.2) \quad \begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Y/X \\ \downarrow f & & \downarrow g & & \downarrow h \\ X' & \longrightarrow & Y' & \longrightarrow & Y'/X' \end{array}$$

is a map of cofiber sequences (i.e., short exact sequences in E) and the maps $X \rightarrow X'$ and $Y/X \rightarrow Y'/X'$ are E -stable equivalences. Then, for any object M of C , we have the commutative diagram with exact columns

$$\begin{array}{ccc} \text{Ext}_{C/E}^1(X', M) & \xrightarrow{\cong} & \text{Ext}_{C/E}^1(X, M) \\ \downarrow & & \downarrow \\ \text{Ext}_{C/E}^2(Y'/X', M) & \xrightarrow{\cong} & \text{Ext}_{C/E}^2(Y/X, M) \\ \downarrow & & \downarrow \\ \text{Ext}_{C/E}^2(Y', M) & \longrightarrow & \text{Ext}_{C/E}^2(Y, M) \\ \downarrow & & \downarrow \\ \text{Ext}_{C/E}^2(X', M) & \xrightarrow{\cong} & \text{Ext}_{C/E}^2(X, M) \\ \downarrow & & \downarrow \\ \text{Ext}_{C/E}^3(Y'/X', M) & \xrightarrow{\cong} & \text{Ext}_{C/E}^3(Y/X, M). \end{array}$$

The horizontal maps marked as isomorphisms are isomorphisms because an E -stable equivalence $A \rightarrow B$ induces a natural equivalence of functors $\text{Ext}_{C/E}^i(B, -) \xrightarrow{\cong} \text{Ext}_{C/E}^i(A, -)$ for all $i \geq 1$; this is Lemma 3.6 of [11]. By the Five Lemma, we then

have a natural isomorphism of functors $\text{Ext}_{C/E}^2(Y', -) \xrightarrow{\cong} \text{Ext}_{C/E}^2(Y, -)$. But since every E -projective is E -injective, this natural transformation being a natural isomorphism implies that $Y \rightarrow Y'$ is an E -stable equivalence; this is Lemma 4.2 of [11]. Hence the extension axiom is satisfied if $E = F$. Now if we do not have $E = F$ but instead $F \subseteq E$, then the extension axiom remains satisfied, as we have fewer diagrams to check the extension axiom for.

□

2.3. Pushing forward and pulling back allowable classes. In this subsection we include a few definitions (and basic properties of these definitions) which are, as far as we know, new. They concern the sense in which an allowable class can be “pushed forward” or “pulled back” along an additive functor between abelian categories.

Definition 2.3.1. *Let C, \mathcal{D} be abelian categories and let $r : \mathcal{D} \rightarrow C$ be an additive functor. Suppose E is an allowable class in \mathcal{D} . Let r_*E be the allowable class in C consisting of all short exact sequences in C isomorphic to short exact sequences of the form*

$$rX \xrightarrow{rf} rY \xrightarrow{rg} rZ$$

where

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a short exact sequence in E . We call r_*E the push-forward of E along r .

If the functor r has a left adjoint—e.g. if \mathcal{D} is a reflective subcategory of C —then one can be very explicit about the projectives and stable equivalences relative to r_*E :

Proposition 2.3.2. *Let C, \mathcal{D}, E, r be as in Def. 2.3.1. Suppose r has a left adjoint i . Then an object X of C is r_*E -projective if iX is E -projective.*

*If we additionally assume that r is right exact, then an object X of C is r_*E -projective if and only if iX is E -projective.*

Proof. Suppose X is an object of C such that iX is E -projective. Suppose that $M \rightarrow N$ is an r_*E -epimorphism in C . Then $M \rightarrow N$ is isomorphic to rf for some E -epimorphism $M' \xrightarrow{f} N'$ in \mathcal{D} . For any map $X \rightarrow N'$, the adjoint map $iX \rightarrow N'$ fits into the diagram

$$(2.3) \quad \begin{array}{ccc} iX & & \\ \downarrow & \searrow & \\ M' & \xrightarrow{f} & N' \end{array}$$

and a dotted arrow making the diagram commute exists by virtue of iX being E -projective. So in the adjoint diagram

$$(2.4) \quad \begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ rM' & \xrightarrow{rf} & rN' \end{array}$$

a dotted arrow making the diagram commute exists. Hence any map $X \rightarrow N$ lifts over any r_*E -epimorphism to N , i.e., X is r_*E -projective.

Now suppose r is right exact, X is an r_*E -projective object of C , and $f : M' \rightarrow N'$ an E -epimorphism in \mathcal{D} . Then any map $iX \rightarrow N'$ in \mathcal{D} admits a lift as in diagram 2.3 and hence an adjoint lift as in diagram 2.4 (the assumption that r is right exact is necessary here in order to know that rf is an epimorphism). Hence iX is E -projective. □

Proposition 2.3.3. *Let $\mathcal{C}, \mathcal{D}, E, r, i$ be as in Prop. 2.3.2. Assume r is exact. If two morphisms $f, g : X \rightarrow Y$ in \mathcal{C} are r_*E -stably equivalent, then $if, ig : iX \rightarrow iY$ are E -stably equivalent.*

*If we assume additionally that r is full and faithful, then two morphisms $f, g : X \rightarrow Y$ in \mathcal{C} are r_*E -stably equivalent if and only if $if, ig : iX \rightarrow iY$ are E -stably equivalent.*

Proof. Suppose $f : X \rightarrow Y$ is a morphism in \mathcal{C} which is r_*E -stably equivalent to the zero morphism. Then f factors as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow a \quad \nearrow b & \\ & P & \end{array}$$

for some r_*E -projective P . Hence

$$\begin{array}{ccc} iX & \xrightarrow{if} & iY \\ & \searrow ia \quad \nearrow ib & \\ & iP & \end{array}$$

commutes and, since iP is E -projective, if is E -stably equivalent to zero.

If, furthermore, r is full and faithful, then the counit map $i \circ r \rightarrow \text{id}_{\mathcal{D}}$ is an isomorphism (see e.g. [7] for this fact). So if

$$\begin{array}{ccc} iX & \xrightarrow{if} & iY \\ & \searrow c \quad \nearrow d & \\ & P' & \end{array}$$

is a diagram in \mathcal{D} with P' an E -projective, then the adjoint diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow rc \quad \nearrow rd & \\ & rP' & \end{array}$$

expresses f as factoring through rP' , and since $irP' \cong P'$ is E -projective, rP' is r_*E -projective by Prop. 2.3.2. So under these conditions, the fact that if is E -stably equivalent to zero implies that f is r_*E -stably equivalent to zero. Now two maps f, g are stably equivalent if and only if $f - g$ is stably equivalent to zero; so we have proven our claims about when two maps are E -stably or r_*E -stably equivalent. \square

Proposition 2.3.4. *Let $\mathcal{C}, \mathcal{D}, E, r, i$ be as in Prop. 2.3.2. Assume r is left exact. If $f : X \rightarrow Y$ is an r_*E -stable equivalence, then $if : iX \rightarrow iY$ is an E -stable equivalence.*

*If we assume additionally that r is full and faithful, then a morphism $f : X \rightarrow Y$ in \mathcal{C} is an r_*E -stable equivalence if and only if $if : iX \rightarrow iY$ is an E -stable equivalence.*

Proof. Suppose $f : X \rightarrow Y$ is an r_*E -stable equivalence. Let $\tilde{f} : Y \rightarrow X$ be a map such that $(\tilde{f} \circ f) - \text{id}_X$ factors through an r_*E -projective P_X and $(f \circ \tilde{f}) - \text{id}_Y$ factors through an r_*E -projective P_Y . Then $(i\tilde{f} \circ if) - \text{id}_{iX}$ factors through the E -projective iP_X , and $(if \circ i\tilde{f}) - \text{id}_{iY}$ factors through the E -projective iP_Y , so if is an E -stable equivalence. (That iP_Y is E -projective is due to Prop. 2.3.2.) Again, if r is full and faithful then the counit map $i \circ r \rightarrow \text{id}_{\mathcal{D}}$ is an isomorphism, so in that case every factorization in \mathcal{D} of a map

through an E -projective is isomorphic to the image under i of a factorization in C of a map through an r_*E -projective, hence under these conditions if being an E -stable equivalence implies that f is an r_*E -stable equivalence. \square

One can also pull back an allowable class along a functor:

Definition 2.3.5. Let C, \mathcal{D} be abelian categories and let $i : C \rightarrow \mathcal{D}$ be an additive functor. Suppose E is an allowable class in \mathcal{D} . Let i^*E be the allowable class in C consisting of all short exact sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in C such that

$$iX \xrightarrow{if} iY \xrightarrow{ig} iZ$$

is short exact sequence in E . We call i^*E the pull-back of E along i .

Proposition 2.3.6. Let E, i be as in Def. 2.3.5. If E has retractile monics, then so does i^*E .

Proof. Suppose E has retractile monics and $g \circ f$ is a monic in i^*E . Then $i(g \circ f)$ is a monic in E , so if is a monic in E , so f is a monic in i^*E . So i^*E has retractile monics. \square

We will make heavy use of push-back and pull-forward allowable classes in the next section.

Corollary 2.3.7. Let C, \mathcal{D}, E, i be as in Def. 2.3.5. Suppose i has an exact, full, faithful right adjoint r , and suppose E has retractile monics and sectile epics, suppose that \mathcal{D} has enough E -projectives and enough E -injectives, and suppose that every E -projective is E -injective.

Then C admits the structure of a Waldhausen category in which the i^*E -monomorphisms are the cofibrations and the r_*E -stable equivalences are the weak equivalences. This Waldhausen category satisfies the saturation axiom and the extension axiom.

Proof. Since E has retractile monics, by Prop. 2.3.6, so does i^*E . Then, by Lemma 2.2.3, the i^*E -monomorphisms are closed under composition. By Lemma 2.2.4, a composite of r_*E -stable equivalences is an r_*E -stable equivalence, so weak equivalences are closed under composition as well.

Now we check the axioms:

- Axioms (Cof 1), (Cof 2), and (Weq 1) are immediate.
- Axiom (Cof 3) follows from Lemma 2.2.2 together with Prop. 2.3.6.
- Axiom (Weq 2) takes some more work. Let

$$\begin{array}{ccccc} Y & \longleftarrow & X & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longleftarrow & X' & \longrightarrow & Z' \end{array}$$

be a commutative diagram in C in which the maps $X \rightarrow Y$ and $X' \rightarrow Y'$ are i^*E -monomorphisms and all three vertical maps are r_*E -stable equivalences. Then, by Prop. 2.3.3 and the definition of i^*E -monomorphisms, the diagram

$$\begin{array}{ccccc} iY & \longleftarrow & iX & \longrightarrow & iZ \\ \downarrow & & \downarrow & & \downarrow \\ iY' & \longleftarrow & iX' & \longrightarrow & iZ' \end{array}$$

is commutative in \mathcal{D} , the maps $iX \rightarrow iY$ and $iX' \rightarrow iY'$ are E -monomorphisms, and the vertical maps are E -stable equivalences. So, by Prop.-Def. 2.2.5, \mathcal{D} admits a Waldhausen category structure in which E -monomorphisms are cofibrations and E -stable equivalences are weak equivalences, so the pushout map

$$(2.5) \quad iY \coprod_{iX} iZ \rightarrow iY' \coprod_{iX'} iZ'$$

is an E -stable equivalence in \mathcal{D} .

Now we have not assumed i to be left exact, so in general it will not preserve colimits, but since $X \rightarrow Y$ is assumed to be an i^*E -monomorphism, i sends the short exact sequence

$$0 \rightarrow X \rightarrow Y \oplus Z \rightarrow Y \coprod_X Z \rightarrow 0$$

to the short exact sequence

$$0 \rightarrow iX \rightarrow iY \oplus iZ \rightarrow i(Y \coprod_X Z) \rightarrow 0,$$

i.e., $i(Y \coprod_X Z) \cong iY \coprod_{iX} iZ$, and similarly, $i(Y' \coprod_{X'} Z') \cong iY' \coprod_{iX'} iZ'$. So map 2.5 is, up to isomorphism, the map

$$i(Y \coprod_X Z) \rightarrow i(Y' \coprod_{X'} Z'),$$

which we now know to be an E -stable equivalence. So, by Prop. 2.3.3,

$$Y \coprod_X Z \rightarrow Y' \coprod_{X'} Z'$$

is an r_*E -stable equivalence. So axiom (Weq 2) is satisfied.

- The saturation axiom follows from Lemma 2.2.4 as well the observation that stable equivalences have up-to-equivalence inverses, as in Prop.-Def. 2.2.5.
- If we were to let cofibrations be i^*E -monomorphisms and weak equivalences be i^*E -stable equivalences, then the extension axiom is satisfied by the same argument as in Prop.-Def. 2.2.5. In our situation, with cofibrations the r_*E -monomorphisms instead, there are strictly fewer diagrams for which the extension diagram needs to be satisfied, so it remains satisfied.

□

3. ABSOLUTE AND RELATIVE QUASI-FROBENIUS CONDITIONS.

3.1. Definitions. To our knowledge these definitions are all new. They are variants of the condition that every object embeds in a projective object, which Faith and Walker showed (see Thm. 1.0.1) to be equivalent, for categories of modules over a ring, to the ring being quasi-Frobenius.

Definition 3.1.1. • Let \mathcal{C} be an abelian category, E, F a pair of allowable classes in \mathcal{C} . We say that \mathcal{C} is cone-Frobenius relative to E, F if, for any object X of \mathcal{C} , there exists an F -monomorphism from X to an E -projective object of \mathcal{C} .

We say that \mathcal{C} is functorially cone-Frobenius relative to E, F if there exists a functor $J : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $\eta : \text{id}_{\mathcal{C}} \rightarrow J$ such that:

- (1) $J(X)$ is E -projective for every object X of \mathcal{C} ,
- (2) $\eta(X) : X \rightarrow J(X)$ is an F -monomorphism for every object X of \mathcal{C} , and
- (3) if $f : X \rightarrow Y$ is an F -monomorphism then so is $J(f) : J(X) \rightarrow J(Y)$.

We sometimes call the pair J, η a cone functor on C relative to E, F . When E, F are understood from context we simply call J, η a relative cone functor.

- **(The absolute case.)** If $E = F$ is the class of all short exact sequences in C and C is cone-Frobenius relative to E, F , then we simply say that C is cone-Frobenius.
If $E = F$ is the class of all short exact sequences in C and C is functorially cone-Frobenius relative to E, F , then we simply say that C is functorially cone-Frobenius.

The idea behind this definition is that the F -monomorphism and E -projective object together are a kind of “mapping cone,” in the sense of homotopy theory, on X : an embedding into a contractible object.

We also want to consider a relative form of the cone-Frobenius condition:

Definition 3.1.2. • If C, C' are abelian categories and $i : C' \rightarrow C$ is an additive functor with right adjoint r , let E denote the allowable class of all short exact sequences in C and let E' the allowable class of all short exact sequences in C' . Then we say that C' is relatively quasi-Frobenius over C (resp. functorially relatively quasi-Frobenius over C) if C' is cone-Frobenius relative to i^*E, r_*E (resp. functorially cone-Frobenius relative to i^*E, r_*E).

The idea here is that, in C' , any object embeds into some other object in such a way that, once one applies i , one gets a mapping cone (an E -monomorphic embedding into a contractible, i.e., E -projective, object) in C . If the reader is wondering whether such a thing is really more than a cone functor on C' or a cone functor on C , the reader might try letting C and C' be module categories over rings, and let i be the base-change/extension-of-scalars functor induced by a *surjection* of rings. This is the example we will ultimately be most interested in, in Thm. 4.2.4 and Cor. 4.2.7 for example. The fact that ring surjections are rarely flat and hence that i is not generally left exact, i.e., i does not generally preserve monomorphisms, means that most attempts one might make to produce a mapping cone on C' do not give mapping cones on C after applying i . So the relative Frobenius conditions really are expressing something nontrivial.

Now the Faith-Walker Theorem, stated above as Thm. 1.0.1, can be restated using our definitions: *a ring R is quasi-Frobenius if and only if the category of R -modules is cone-Frobenius.*

3.2. Existence of cylinder functors. Now we show the equivalence of the functorial cone-Frobenius condition with the existence of cylinder functors satisfying the cylinder axioms.

Theorem 3.2.1. *Let C, E, F be as in Prop.-Def. 2.2.5. Then the Waldhausen category $\mathcal{C}_{E-we}^{F-cof}$ admits a cylinder functor satisfying the cylinder axiom if and only if C is functorially cone-Frobenius relative to E, F .*

Proof. If $\mathcal{C}_{E-we}^{F-cof}$ has a cylinder functor satisfying the cylinder axiom, then the cylinder functor I sends, for any object X of C , the map $X \rightarrow \text{pt.}$ to the diagram

$$\begin{array}{ccc} X & \xrightarrow{F-cof} & I(X) \longleftarrow \text{pt.} \\ & \searrow & \downarrow E-we \swarrow \\ & & \text{pt.} \end{array}$$

where the map marked $F - cof$ is an F -monic and the map marked $E - we$ is an E -stable equivalence. But for an object's map to the zero object to be an E -stable equivalence,

this is equivalent to that object being an E -projective. Furthermore, if $f : X \rightarrow Y$ is any F -monomorphism in \mathcal{C} , then we have the commutative diagram

$$\begin{array}{ccc} X \amalg \text{pt.} & \longrightarrow & I(X \amalg \text{pt.}) \\ \downarrow f \amalg \text{id}_{\text{pt.}} & & \downarrow I(f \amalg \text{id}_{\text{pt.}}) \\ Y \amalg \text{pt.} & \longrightarrow & I(Y \amalg \text{pt.}) \end{array}$$

and condition (Cyl 1) in the definition of a cylinder functor requires that $I(f \amalg \text{id}_{\text{pt.}})$ be an F -monomorphism as well. This completes one direction of the proof: $X \mapsto I(X \rightarrow \text{pt.})$ is a relative cone functor.

Now suppose we have a functor J and natural transformation e as in the definition of a relative cone functor in Def. 3.1.1. We claim that the functor sending any map $X \xrightarrow{f} Y$ to the diagram

$$\begin{array}{ccccc} X & \xrightarrow{(f, e(X))} & Y \oplus J(X) & \xleftarrow{(\text{id}, 0)} & Y \\ & \searrow f & \downarrow \pi_Y & \swarrow \text{id} & \\ & & Y & & \end{array},$$

where π_Y is projection to Y , is a cylinder functor on \mathcal{C} satisfying the cylinder axiom. We check Waldhausen's conditions from Def. 2.1.11. Condition (Cyl 2) is immediate, and the cylinder axiom follows from the projection $J(X) \oplus Y \rightarrow \text{pt.} \oplus Y \cong Y$ being a direct sum of E -stable equivalences, hence itself an E -stable equivalence.

We handle condition (Cyl 1) as follows: if $X' \rightarrow X$ and $Y' \rightarrow Y$ are F -monomorphisms, then the direct sum $J(X') \oplus Y' \rightarrow J(X) \oplus Y$ is an F -monomorphism by the assumption that J sends F -monomorphisms to F -monomorphisms. We also note that, since $J(X')$, $J(X)$ are E -projective, the projections $J(X') \rightarrow \text{pt.}$ and $J(X) \rightarrow \text{pt.}$ are E -stable equivalences, and Lemma 2.2.4 gives us that the composite of either one with an E -stable inverse of the other is an E -stable equivalence between $J(X)$ and $J(X')$. So $J(X') \oplus Y' \rightarrow J(X) \oplus Y$ is an E -stable equivalence. So condition (Cyl 1) holds. \square

Corollary 3.2.2. *Let \mathcal{C} be an abelian category, and let E be a quasi-Frobenius allowable class in \mathcal{C} . Suppose \mathcal{C} has functorially enough E -injectives. Then $\mathcal{C}_{E-\text{we}}^{E-\text{cof}}$ has a cylinder functor satisfying the cylinder axiom.*

Proof. We claim that the functor I given by \mathcal{C} having functorially enough E -injectives is in fact a relative cone functor, since E is quasi-Frobenius and hence $I(X)$ being E -injective means that $I(X)$ is also E -projective. The axioms from the definition of a relative cone functor in Def. 3.1.1 are immediate from the definition of \mathcal{C} having functorially enough E -injectives. Now Thm. 3.2.1 implies that, since $\mathcal{C}_{E-\text{we}}^{E-\text{cof}}$ has a relative cone functor, it has a cylinder functor satisfying the cylinder axiom. \square

3.3. Compatibility of cylinder functors, and using the Fibration Theorem. We will have to make three separate uses of Waldhausen's Approximation Theorem, from [14], in order to prove our main theorem, Thm. 3.3.5. For the sake of clarity we state the Approximation Theorem there:

Theorem 3.3.1. Approximation Theorem (Waldhausen). *Suppose $\mathcal{C}, \mathcal{C}'$ are Waldhausen categories satisfying the saturation axiom. Suppose \mathcal{C}' admits a cylinder functor satisfying the cylinder axiom. Suppose that $F : \mathcal{C}' \rightarrow \mathcal{C}$ is an exact functor satisfying the following conditions:*

- **(App 1)** A map f in \mathcal{C}' is a weak equivalence if and only if Ff is a weak equivalence in \mathcal{C} .
- **(App 2)** Given any map $g : FX \rightarrow Y$ in \mathcal{C} there exists a cofibration $h : X \rightarrow X'$ in \mathcal{C}' and a weak equivalence $g' : FX' \rightarrow Y$ in \mathcal{C} such that the diagram

$$\begin{array}{ccc} FX & \xrightarrow{g} & Y \\ \downarrow Fh & \nearrow g' & \\ FX' & & \end{array}$$

commutes.

Then the maps of spaces

$$|w\mathcal{C}'| \rightarrow |w\mathcal{C}|$$

and

$$|wS.\mathcal{C}'| \rightarrow |wS.\mathcal{C}|$$

induced by F are homotopy equivalences.

Now we prove a technical lemma we will need later on. It states, very very roughly, that “push-forward of allowable classes induces a K -theory equivalence”:

Lemma 3.3.2. *Let $\mathcal{C}, \mathcal{D}, E, i, r$ be as in Cor. 2.3.7. Let F be an allowable class in \mathcal{D} with $F \subseteq E$, and let F' be an allowable class in \mathcal{C} with $F' \subseteq i^*E$. Suppose that F, F' both have retractile monics, suppose that \mathcal{D} has enough F -injectives and \mathcal{C} has enough F' -injectives, suppose that \mathcal{C} is functorially cone-Frobenius relative to r_*E, i^*F , and suppose that i sends any F' -monomorphism in \mathcal{C} to an F -monomorphism in \mathcal{D} . Then the induced maps*

$$|wi| : |w\mathcal{C}_{r_*E-we}^{F'-cof}| \rightarrow |w\mathcal{D}_{E-we}^{F-cof}|$$

and

$$|wS.i| : |wS.\mathcal{C}_{r_*E-we}^{F'-cof}| \rightarrow |wS.\mathcal{D}_{E-we}^{F-cof}|$$

are homotopy equivalences.

In particular, the induced map of K -theory groups

$$K_n(i) : K_n(\mathcal{C}_{r_*E-we}^{F'-cof}) \rightarrow K_n(\mathcal{D}_{E-we}^{F-cof})$$

is an isomorphism for every $n \geq 0$.

Proof. This is an application of Waldhausen’s Approximation Theorem, reproduced above as Thm. 3.3.1. We must know that $\mathcal{C}_{r_*E-we}^{F'-cof}, \mathcal{D}_{E-we}^{F-cof}$ both satisfy the saturation axiom, and that $\mathcal{C}_{r_*E-we}^{F'-cof}$ has a cylinder functor satisfying the cylinder axiom. These conditions are all satisfied by the Waldhausen category structures defined in Def.-Prop. 2.2.5 and the cylinder functors arising from Thm. 3.2.1. We must check Waldhausen’s two remaining conditions. Condition (App 1) follows immediately from Prop. 2.3.4: r_*E -stable equivalences are precisely the maps which are sent by i to E -stable equivalences. For condition (App 2), let $g : iX \rightarrow Y$ be a map in \mathcal{D} , and let (J, e) be a relative cone functor on $\mathcal{C}_{r_*E-we}^{F'-cof}$. Let a be the map $a : X \rightarrow J(X) \oplus rY$ given by the matrix $\begin{bmatrix} e(X) \\ g^b \end{bmatrix}$, where $g^b : X \rightarrow rY$ is the adjoint map to $g : iX \rightarrow Y$. We write $\pi_1 : J(X) \oplus rY \rightarrow J(X)$ and $\pi_2 : J(X) \oplus rY \rightarrow rY$ for the projection to the first and second factors, respectively. Then since $\pi_1 \circ a$ is the identity map on $J(X)$, which is an F' -monic, and F' has retractile monics, a is also an F' -monic, i.e., a cofibration in $\mathcal{C}_{r_*E-we}^{F'-cof}$. Furthermore, the adjoint map $\pi_2^\# : iJ(X) \oplus irN \rightarrow N$ to π_2 is an

E -stable equivalence, since $J(X)$ is an r_*E -projective and hence $iJ(X)$ is an E -projective, and the counit map $irN \rightarrow N$ is an isomorphism. Hence condition (App 2) is satisfied. \square

Lemma 3.3.2 is not directly very useful on its own, but it plays an important role in the proof of Thm. 3.3.5. Typically one has \mathcal{C}, \mathcal{D} some categories of finitely-generated modules (over a ring, scheme, stack, etc.), and $i : \mathcal{C} \rightarrow \mathcal{D}$ is a base-change functor. If one is working in the “absolute” case, i.e., F' is the allowable class consisting of all short exact sequences in \mathcal{C} and F consists of all short exact sequences in \mathcal{D} , then the assumption made in Lemma 3.3.2 that the functor i preserves cofibrations only holds when the base-change is *flat*. Certainly this happens sometimes, but the most interesting cases, where we will want to be able to make K -theory and stable G -theory computations, are when the base-change functor is not flat.

On the other hand, we can let $F' = i^*F$ to force the functor i to preserve cofibrations, so that Lemma 3.3.2 is applicable, and then we can try to use *another* Approximation Theorem argument to prove that the Waldhausen categories $\mathcal{C}_{r_*E-we}^{F'-cof}$ and $\mathcal{C}_{r_*E-we}^{i^*F-cof}$ have the same K -theory. This leads to the rather useful Prop. 3.3.3:

Proposition 3.3.3. *Let $\mathcal{C}, \mathcal{D}, E, F, F', r, i$ be as in Lemma 3.3.2. Suppose $i^*F \subseteq F'$ and suppose that \mathcal{C} is functorially cone-Frobenius relative to r_*E, i^*F . Then the maps of spaces*

$$\begin{array}{ccc} & & |wS.\mathcal{C}_{r_*E-we}^{F'-cof}| \\ & \nearrow & \\ |wS.\mathcal{C}_{r_*E-we}^{i^*F-cof}| & & \\ & \searrow \scriptstyle |wS.i| & \\ & & |wS.\mathcal{D}_{E-we}^{F-cof}| \end{array}$$

are both homotopy equivalences.

Proof. Prop. 2.3.6 gives us that F having retractile monics implies that i^*F has retractile monics, so now Lemma 3.3.2 implies that

$$|wi| : |w\mathcal{C}_{r_*E-we}^{i^*F-cof}| \rightarrow |w\mathcal{D}_{E-we}^{F-cof}|$$

and

$$|wS.i| : |wS.\mathcal{C}_{r_*E-we}^{i^*F-cof}| \rightarrow |wS.\mathcal{D}_{E-we}^{F-cof}|$$

are homotopy equivalences. That

$$|w\mathcal{C}_{r_*E-we}^{i^*F-cof}| \rightarrow |w\mathcal{C}_{r_*E-we}^{F'-cof}|$$

and

$$|wS.\mathcal{C}_{r_*E-we}^{i^*F-cof}| \rightarrow |wS.\mathcal{C}_{r_*E-we}^{F'-cof}|$$

are homotopy equivalences is a direct application of Waldhausen’s Approximation Theorem. \square

Prop. 3.3.3 has several nice consequences. Here are a few of them: of a computationally useful nature, one has the long exact sequence of Thm. 3.3.5 and the spectral sequence of its corollary, Cor. 3.3.8. Of a more conceptual nature, one has Cor. 3.3.4, which gives criteria for a base-change functor between abelian categories to induce, up to equivalence, a localization map on K -theory spaces; that is, a map induced by simply expanding the class of weak equivalences on a Waldhausen category.

Corollary 3.3.4. (When is a map of Waldhausen categories equivalent to one induced by a localization?) *Let $\mathcal{C}, \mathcal{D}, i, r, E, F, F'$ be as in Prop. 3.3.3. Suppose E' is an allowable class in \mathcal{C} such that, if f is an E' -projective, then $i(f)$ is an E -projective. Then the map of K -theory spaces induced by i*

$$|wS \cdot \mathcal{C}_{E'-we}^{F'-cof}| \xrightarrow{|wS \cdot i|} |wS \cdot \mathcal{D}_{E-we}^{F-cof}|$$

is, up to equivalence, just the map induced by the change of the class of weak equivalences

$$|wS \cdot \mathcal{C}_{E'-we}^{F'-cof}| \longrightarrow |wS \cdot \mathcal{C}_{r_*E-we}^{F-cof}|,$$

i.e., the map $|wS \cdot i|$ is, up to equivalence, induced by a localization.

Proof. This follows immediately from Prop. 3.3.3 as soon as we know that the identity functor on \mathcal{C} is an exact (in the Waldhausen category sense, i.e., sending cofibrations to cofibrations and sending weak equivalences to weak equivalences) functor $\mathcal{C}_{E'-we}^{F'-cof} \longrightarrow \mathcal{C}_{r_*E-we}^{F-cof}$. The condition that i sends E' -projectives to E -projectives guarantees that, by Prop. 2.3.2, any E' -projective object is an r_*E -projective object. Hence E' -stable equivalences are also r_*E -stable equivalences. Together with the assumption $i^*F \subseteq F'$, this implies that the identity functor on \mathcal{C} induces the desired exact functor $\mathcal{C}_{E'-we}^{F'-cof} \longrightarrow \mathcal{C}_{r_*E-we}^{F-cof}$. \square

The next theorem is what we consider the main result of this paper:

Theorem 3.3.5. *Let \mathcal{C}, \mathcal{D} be abelian categories, $r : \mathcal{D} \rightarrow \mathcal{C}$ an exact full faithful functor with left adjoint i . Let E be an allowable class in \mathcal{D} with retractile monics and sectile epics and such that \mathcal{D} has enough E -injectives and enough E -projectives. Suppose every E -projective is E -injective. Let E', F' be allowable classes in \mathcal{C} such that:*

- $F' \subseteq i^*E \subseteq E'$,
- F' has retractile monics,
- i sends E' -projectives to E -projectives,
- \mathcal{C} has enough F' -injectives,
- \mathcal{C} is functorially cone-Frobenius relative to E', F' , and
- \mathcal{C} is functorially relatively quasi-Frobenius over \mathcal{D} .

Then the homotopy fibre of the map

$$|wS \cdot \mathcal{C}_{E'-we}^{F'-cof}| \rightarrow |wS \cdot \mathcal{D}_{E-we}^{E-cof}|$$

is the space $|wS \cdot X|$, where X is the full sub-Waldhausen-category of $\mathcal{C}_{E'}^{F'}$ generated by the objects X such that iX is E -projective. Hence we get a long exact sequence in K -groups

$$\cdots \rightarrow K_{n+1}(\mathcal{D}_{E-we}^{E-cof}) \rightarrow K_n X \rightarrow K_n(\mathcal{C}_{E'-we}^{F'-cof}) \rightarrow K_n(\mathcal{D}_{E-we}^{E-cof}) \rightarrow K_{n-1} X \rightarrow \cdots$$

Proof. Thm. 3.2.1 and Def.-Prop. 2.2.5 together imply that $\mathcal{C}_{E'-we}^{F'-cof}$ has a cylinder functor and that the same cylinder functor is a cylinder functor on $\mathcal{C}_{r_*E-we}^{F'-cof}$ satisfying the saturation and extension axioms. So Waldhausen's Fibration Theorem (see Thm. 2.1.14) implies that

$$|wS \cdot \mathcal{C}_{E'-we}^{F'-cof}| \rightarrow |wS \cdot \mathcal{C}_{r_*E-we}^{F'-cof}|$$

has the stated homotopy fibre. \square

An important special case of Thm. 3.3.5 is the case where E' is the allowable class of *all* short exact sequences in \mathcal{C} and E is the allowable class of *all* short exact sequences in \mathcal{D} . In this setting, the E -projectives in \mathcal{C} are the ordinary projective objects in \mathcal{C} , and the E' -projectives in \mathcal{D} are the ordinary projective objects in \mathcal{D} , so the associated K -theory groups are the *stable G -theory groups* of \mathcal{C} and \mathcal{D} :

Definition 3.3.6. Suppose \mathcal{C} is an abelian category in which every projective object is injective. The stable G -theory groups of \mathcal{C} are the groups

$$(G_{st})_n(\mathcal{C}) \cong \pi_n \Omega \left|_{\mathcal{WS} \cdot \mathcal{C}_{E-we}^{E-cof}} \right|,$$

where E is the allowable class of all short exact sequences in \mathcal{C} .

In this special case, Thm. 3.3.5 gives us:

Corollary 3.3.7. Let \mathcal{C}, \mathcal{D} be abelian categories with enough projectives and enough injectives and in which every projective object is injective. Let $r : \mathcal{D} \rightarrow \mathcal{C}$ be a right exact full faithful additive functor with left adjoint i . Suppose that \mathcal{C} is functorially cone-Frobenius and suppose also that \mathcal{C} is functorially relatively quasi-Frobenius over \mathcal{D} . Then we get a long exact sequence in stable G -theory groups

$$\cdots \rightarrow (G_{st})_{n+1}(\mathcal{D}) \rightarrow K_n \mathcal{X} \rightarrow (G_{st})_n(\mathcal{C}) \rightarrow (G_{st})_n(\mathcal{D}) \rightarrow K_{n-1} \mathcal{X} \rightarrow \cdots,$$

where \mathcal{X} is the full sub-Waldhausen-category of \mathcal{C} generated by all objects X such that iX is projective in \mathcal{D} . The cofibrations in \mathcal{X} are the morphisms in \mathcal{X} which are monomorphisms in \mathcal{C} , and the weak equivalences in \mathcal{X} are the morphisms in \mathcal{X} which are stable equivalences in \mathcal{C} .

Corollary 3.3.8. Let \mathcal{C} be an abelian category equipped with a finite filtration by abelian subcategories

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \cdots \subseteq \mathcal{C}_n = \mathcal{C}$$

such that each inclusion functor $\mathcal{C}_j \rightarrow \mathcal{C}_{j+1}$ is right exact and additive and has a left adjoint i_j . Suppose, for all j , that:

- projective objects in \mathcal{C}_j are injective,
- \mathcal{C}_j has enough projectives,
- \mathcal{C}_j is functorially cone-Frobenius
- and \mathcal{C}_{j+1} is functorially relatively quasi-Frobenius over \mathcal{C}_j .

For each $j > 0$, we write \mathcal{X}_j for the full sub-Waldhausen-category of \mathcal{C}_j generated by the objects X such that $i_j(X)$ is weakly equivalent to 0 in \mathcal{C}_{j-1} . We set \mathcal{X}_0 equal to \mathcal{C}_0 . Then there exists a strongly convergent spectral sequence

$$E_1^{p,q} \cong K_p(\mathcal{X}_q) \Rightarrow (G_{st})_p(\mathcal{C}).$$

Here the differentials behave as follows:

$$d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p-1,q+r}.$$

Proof. This is just the homotopy spectral sequence of the tower of fibrations

$$\begin{array}{ccc}
 \Omega |wS \cdot \mathcal{X}_n| & \longrightarrow & \Omega |wS \cdot C_n| \\
 & & \downarrow \\
 \Omega |wS \cdot \mathcal{X}_{n-1}| & \longrightarrow & \Omega |wS \cdot C_{n-1}| \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 \Omega |wS \cdot \mathcal{X}_1| & \longrightarrow & \Omega |wS \cdot C_1| \\
 & & \downarrow \\
 \Omega |wS \cdot \mathcal{X}_0| & \xrightarrow{\cong} & \Omega |wS \cdot C_0|.
 \end{array}$$

One knows that it is a tower of fibrations, i.e., each L -shaped consecutive pair of maps is a homotopy fibre sequence, from Thm 3.3.5. For a reference on the spectral sequence of a tower of fibrations one can consult e.g. [2]. \square

4. APPLICATIONS.

4.1. The relationship between stable G -theory and other G -theories and K -theories.

We recall that stable G -theory was defined in Def. 3.3.6. It sits naturally in a diagram relating it to algebraic K -theory, algebraic G -theory, and the “derived representation groups,” but the relationship between these theories does not seem to be simple enough to permit easy computation of one from the others. We describe this relationship and provide a few comments in Remark 4.1.4 about the computational task of computing one of these theories once one has computed the others.

Now we define some notations we use to describe certain Waldhausen categories associated to an abelian category \mathcal{C} :

Definition 4.1.1. *Suppose \mathcal{C} is an abelian category. We will write:*

- $\mathcal{K}^\oplus(\mathcal{C})$ for the split K -theory category of \mathcal{C} , i.e., the Waldhausen category structure on the full subcategory generated by the projective objects of \mathcal{C} , where cofibrations are split inclusions and weak equivalences are isomorphisms.
- $\mathcal{K}(\mathcal{C})$ for the nonsplit K -theory category of \mathcal{C} , i.e., the Waldhausen category structure on the full subcategory generated by the projective objects of \mathcal{C} , where cofibrations are inclusions and weak equivalences are isomorphisms.
- $\mathcal{G}^\oplus(\mathcal{C})$ for the split G -theory category of \mathcal{C} , i.e., the Waldhausen category structure on \mathcal{C} where cofibrations are split inclusions and weak equivalences are isomorphisms.
- $\mathcal{G}(\mathcal{C})$ for the G -theory category of \mathcal{C} , i.e., the Waldhausen category structure on \mathcal{C} where cofibrations are inclusions and weak equivalences are isomorphisms.
- $\mathcal{G}_{st}^\oplus(\mathcal{C})$ for the stable split G -theory category of \mathcal{C} , i.e., the Waldhausen category structure on \mathcal{C} where cofibrations are split inclusions and weak equivalences are stable equivalences. (In the absolute sense, i.e., E -stable equivalences where E is the allowable class of all short exact sequences in \mathcal{C} .)
- $\mathcal{G}_{st}(\mathcal{C})$ for the stable G -theory category of \mathcal{C} , i.e., the Waldhausen category structure on \mathcal{C} where cofibrations are inclusions and weak equivalences are stable

equivalences. (In the absolute sense, i.e., E -stable equivalences where E is the allowable class of all short exact sequences in C .)

We note that, if R is a ring and C is the category of finitely generated R -modules, then

$$\pi_n \Omega |wS \cdot \mathcal{K}^\oplus(C)| \cong K_n(R),$$

i.e., the Waldhausen K -theory of the split K -theory Waldhausen category recovers the classical algebraic K -theory of R . Meanwhile,

$$\pi_n \Omega |wS \cdot \mathcal{G}(C)| \cong G_n(R),$$

i.e., the Waldhausen G -theory of the (nonsplit) G -theory Waldhausen category recovers the classical algebraic G -theory of R .

The other theories are more obscure but still meaningful. In degree zero,

$$\pi_0 \Omega |wS \cdot \mathcal{G}^\oplus(C)| \cong \text{Rep}(R),$$

the representation group (actually ring, under tensor product; but we have not discussed any multiplicative structures on our Waldhausen categories, which is another subject entirely) of R —that is, the Grothendieck group completion of the monoid of isomorphism classes of finitely-generated R -modules. So we sometimes regard the split G -theory as the “derived representation theory” and the groups $\pi_* \Omega |wS \cdot \mathcal{G}^\oplus(C)|$ as the “derived representation groups” of R .

In degree zero,

$$\pi_0 \Omega |wS \cdot \mathcal{G}_{st}^\oplus(C)| \cong \text{StableRep}(R),$$

the stable representation group of R —that is, the Grothendieck group completion of the monoid of stable equivalence classes of finitely-generated R -modules. So we sometimes regard the split stable G -theory as the “derived stable representation theory” and the groups $\pi_* \Omega |wS \cdot \mathcal{G}_{st}^\oplus(C)|$ as the “derived stable representation groups” of R .

Finally, the results of this paper, especially Thm. 3.3.5 and Cor. 3.3.8, are really about the stable G -theory groups

$$\pi_* \Omega |wS \cdot \mathcal{G}_{st}(C)| \cong (G_{st})_*(C),$$

as defined in Def. 3.3.6. In degree zero, (nonsplit) stable G -theory is

$$\pi_0 \Omega |wS \cdot \mathcal{G}_{st}(C)| \cong \text{StableRep}(R)/A,$$

the stable representation group modulo the subgroup A generated by all elements of the form $L - M + N$ where

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is a short exact sequence in C .

Proposition 4.1.2. *For any abelian category C , we have a commutative diagram of topological spaces*

$$(4.6) \quad \begin{array}{ccccc} |wS \cdot \mathcal{K}^\oplus(C)| & \longrightarrow & |wS \cdot \mathcal{G}^\oplus(C)| & \longrightarrow & |wS \cdot \mathcal{G}_{st}^\oplus(C)| \\ \downarrow & & \downarrow & & \downarrow \\ |wS \cdot \mathcal{K}(C)| & \longrightarrow & |wS \cdot \mathcal{G}(C)| & \longrightarrow & |wS \cdot \mathcal{G}_{st}(C)| \end{array}$$

in which the horizontal composites are nulhomotopic.

Suppose further that every projective object in C is also injective. Then the map

$$|wS \cdot \mathcal{K}^\oplus(C)| \rightarrow |wS \cdot \mathcal{K}(C)|$$

in the above diagram is a homotopy equivalence.

Proof. First, when every projective object in \mathcal{C} is also injective, then any injective map between projective objects splits, by the universal property of an injective object; so injections and split injections coincide in the categories of projective objects in \mathcal{C} , so the functor $\mathcal{K}^{\oplus}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{C})$ is an isomorphism of Waldhausen categories.

Everything else here is a consequence of elementary facts from [14]. \square

We have a negative result:

Proposition 4.1.3. *The horizontal composite*

$$|wS \cdot \mathcal{K}(\mathcal{C})| \rightarrow |wS \cdot \mathcal{G}(\mathcal{C})| \rightarrow |wS \cdot \mathcal{G}_{st}(\mathcal{C})|$$

is not always of the homotopy type of a fibration. Specifically, if it is a fibration for $\mathcal{C} \cong \text{fgMod}(\mathbb{F}_2[x]/x^2)$, then it is not a fibration for $\mathcal{C} \cong \text{fgMod}(\mathbb{F}_2[x]/x^4)$.

Proof. If R is a ring, we will $\mathcal{K}(R)$, $\mathcal{G}(R)$, etc., as shorthand for $\mathcal{K}(\text{fgMod}(R))$, $\mathcal{G}(\text{fgMod}(R))$, etc. Suppose

$$(4.7) \quad |wS \cdot \mathcal{K}(\mathbb{F}_2[x]/x^2)| \rightarrow |wS \cdot \mathcal{G}(\mathbb{F}_2[x]/x^2)| \rightarrow |wS \cdot \mathcal{G}_{st}(\mathbb{F}_2[x]/x^2)|$$

is of the homotopy type of a fibre sequence. Since the map

$$\mathbb{Z} \cong K_0(\mathbb{F}_2[x]/x^2) \rightarrow G_0(\mathbb{F}_2[x]/x^2) \cong \mathbb{Z}$$

is injective (in fact it is multiplication by two), the connecting map $(G_{st})_1(\mathbb{F}_2[x]/x^2) \rightarrow K_0(\mathbb{F}_2[x]/x^2)$ must be zero, so we have an exact sequence (drawn as a column so it will fit within the margins):

$$\begin{array}{ccc} G_2(\mathbb{F}_2[x]/x^2) & \xrightarrow{\cong} & 0 \\ \downarrow & & \\ (G_{st})_2(\mathbb{F}_2[x]/x^2) & & \\ \downarrow & & \\ K_1(\mathbb{F}_2[x]/x^2) & \xrightarrow{\cong} & (\mathbb{F}_2[x]/x^2)^{\times} \\ \downarrow & & \\ G_1(\mathbb{F}_2[x]/x^2) & \xrightarrow{\cong} & (\mathbb{F}_2)^{\times} \cong 0 \\ \downarrow & & \\ (G_{st})_1(\mathbb{F}_2[x]/x^2) & & \\ \downarrow & & \\ 0 & & \end{array}$$

The isomorphism $G_2(\mathbb{F}_2[x]/x^2) \cong 0$ is due to G -theory being invariant under nilpotent extension, so $G_2(\mathbb{F}_2[x]/x^2) \cong G_2(\mathbb{F}_2) \cong K_2(\mathbb{F}_2) \cong 0$ by Quillen's computation of the algebraic K -theory of finite fields in [9]. The isomorphism $K_1(\mathbb{F}_2[x]/x^2) \cong (\mathbb{F}_2[x]/x^2)^{\times}$ is the identification of K_1 of a commutative local ring, as in [1]. The isomorphism $G_1(\mathbb{F}_2[x]/x^2) \cong 0$ follows from the chain of isomorphisms

$$G_1(\mathbb{F}_2[x]/x^2) \cong G_1(\mathbb{F}_2) \cong K_1(\mathbb{F}_2) \cong (\mathbb{F}_2)^{\times} \cong 0.$$

Hence, if 4.7 is of the homotopy type of a fibre sequence, then $(G_{st})_2(\mathbb{F}_2[x]/x^2) \cong (\mathbb{F}_2[x]/x^2)^\times$. Now suppose

$$(4.8) \quad |wS \cdot \mathcal{K}(\mathbb{F}_2[x]/x^4)| \rightarrow |wS \cdot \mathcal{G}(\mathbb{F}_2[x]/x^4)| \rightarrow |wS \cdot \mathcal{G}_{st}(\mathbb{F}_2[x]/x^4)|$$

is also of the homotopy type of a fibre sequence. Then a similar argument shows that $(G_{st})_2(\mathbb{F}_2[x]/x^4) \cong (\mathbb{F}_2[x]/x^4)^\times$. But then Thm. 4.2.4, together with the connective Hopf algebra extension

$$\mathbb{F}_2[x^2]/x^4 \rightarrow \mathbb{F}_2[x]/x^4 \rightarrow \mathbb{F}_2[x]/x^2,$$

implies that there is an exact sequence

$$(G_{st})_2(\mathbb{F}_2[x]/x^2) \rightarrow (G_{st})_2(\mathbb{F}_2[x]/x^4) \rightarrow (G_{st})_2(\mathbb{F}_2[x]/x^2) \rightarrow 0,$$

i.e., that there exists a surjection $(\mathbb{F}_2[x]/x^4)^\times \rightarrow (\mathbb{F}_2[x]/x^2)^\times$ with kernel admitting a surjection from $(\mathbb{F}_2[x]/x^2)^\times$. But $(\mathbb{F}_2[x]/x^4)^\times \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $1+x$ and $1+x^2+x^3$, and $(\mathbb{F}_2[x]/x^2)^\times \cong \mathbb{Z}/2\mathbb{Z}$, so no such surjection can exist, a contradiction. So 4.7 and 4.8 cannot both be of the homotopy type of a fibre sequence. \square

Remark 4.1.4. Since the composite

$$(4.9) \quad |wS \cdot \mathcal{K}(C)| \rightarrow |wS \cdot \mathcal{G}(C)| \rightarrow |wS \cdot \mathcal{G}_{st}(C)|$$

is nulhomotopic, we get a map from the fiber F_1 of $|wS \cdot \mathcal{K}(C)| \rightarrow |wS \cdot \mathcal{G}(C)|$ to $\Omega |wS \cdot \mathcal{G}_{st}(C)|$, but since 4.9 is not generally of the homotopy type of a fiber sequence, the map $F_1 \rightarrow \Omega |wS \cdot \mathcal{G}_{st}(C)|$ is not generally an equivalence. So one has the beginning of a tower of fibrations

$$\begin{array}{ccc} F_2 & & \\ \downarrow & & \\ F_1 & \longrightarrow & \Omega |wS \cdot \mathcal{G}_{st}(C)| \\ \downarrow & & \\ |wS \cdot \mathcal{K}(C)| & \longrightarrow & |wS \cdot \mathcal{G}(C)|. \end{array}$$

It would be very desirable to find something appropriate and not totally un-computable to map F_2 and the higher fibers to, in order to extend this tower of fibrations. If one can make such choices well enough that the fibers become more connective as one passes up the tower, then the associated spectral sequence would probably be quite computationally valuable: it would give a way to go from algebraic G -theory and stable G -theory, both relatively computable things, to algebraic K -theory, which is generally much, much harder to compute. We don't know how to do this, however.

There is a related computational tool we would really like to have, but don't: we would very much like to be able to compute the homotopy groups of the fiber F^1 of the map

$$|wS \cdot \mathcal{G}_{st}^\oplus(C)| \rightarrow |wS \cdot \mathcal{G}_{st}(C)|.$$

From the associated long exact sequence induced by π_* as well as the identification we have of $(G_{st}^\oplus)_0$ and $(G_{st})_0$, we know (for example) that π_0 surjects on to the subgroup of the stable representation group $\text{StabRep}(C)$ generated by elements of the form $M_0 - M_1 + M_2$ for all short exact sequences $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ in C , but it is hard to say much about $\pi_i(F^1)$ for $i > 0$. Since $(G_{st}^\oplus)_0(C)$ is the stable representation group of C , one should think of $(G_{st}^\oplus)_i(C)$ as the *derived* stable representation groups of C . An identification of $\pi_*(F^1)$ would allow one to pass from a knowledge of $(G_{st})_i(C)$, computable using our localization

results in this paper, to a knowledge of the derived stable representation groups $(G_{st}^\oplus)_i(C)$. This would be valuable.

The problem of identifying F^1 is a special case of the following general problem: when one changes (expands or contracts) the class of cofibrations in a Waldhausen category \mathcal{C} , how does it change the K -theory groups $\pi_*(\Omega|_{wS} \cdot \mathcal{C}|)$? We have an approach to this problem, especially in the situation here of identifying F^1 , in which we use a relative cell decomposition to describe the homology $H_n(|_{wS} \cdot \mathcal{C}|, |_{wS} \cdot \mathcal{C}^\oplus|)$ and then, making use of this homology computation and the relative Hurewicz map, we use an analogue of Serre's method of computing homotopy groups of spheres to work our way up the relative Postnikov tower of the map $|_{wS} \cdot \mathcal{C}^\oplus| \rightarrow |_{wS} \cdot \mathcal{C}|$. This technique is difficult and in any case is beyond the scope of the present paper. We plan to return to it in a later paper.

4.2. Applications to algebras. In this section we will finally apply our results to actual rings and algebras! We will frequently assume that the rings in question are quasi-Frobenius. However, we owe the reader an explanation for why this implies the functorial cone-Frobenius condition. Suppose R is a Noetherian ring, and let U be the injective envelope of the direct sum $\bigoplus R/I$, where I ranges across all right ideals of R . For each right R -module M , let

$$J(M) = \prod_{\text{hom}_R(M, U)} U,$$

and let $\eta(M) : M \rightarrow J(M)$ send m to the map whose component in the factor corresponding to $f \in \text{hom}_R(M, U)$ is $f(m)$. This gives a functorial embedding of every R -module into an injective R -module, and (due to the characterizing property of an injective envelope) J sends monomorphisms to monomorphisms (this construction is due to Bass, and appears in Ex. 5.26 in Lam's book [5]). If R is quasi-Frobenius, then every injective R -module is projective, and J is a cone functor. So the category of finitely-generated modules over any finite-dimensional quasi-Frobenius algebra over a finite field is functorially cone-Frobenius, for example. Throughout this section, whenever we assume that an algebra is over a *finite* field, the only reason we assume finiteness of the field is so that the above construction gives a cone functor on the finitely-generated module category; if one can extend this construction to finitely-generated modules over algebras over more general fields, then one can do away with the finiteness assumption on the field.

Lemma 4.2.1. *Suppose k is a field and R, S are finite-dimensional k -algebras and $R \rightarrow S$ is an algebra map. Suppose that S is free as a right R -module. Suppose M, N are finitely generated R -modules such that $M \otimes_R S \cong N \otimes_R S$ as S -modules. Then $M \cong N$ as R -modules.*

Proof. Since S is free over R , the right R -modules $M \otimes_R S, N \otimes_R S$ (i.e., the modules obtained by base extension and then restriction of scalars) are isomorphic to a finite direct sum of copies of M, N respectively; now the fact that R is a finite-dimensional k -algebra implies that the category of finitely generated R -modules is a Krull-Schmidt category, so knowing that the n -fold direct sum $M \oplus \cdots \oplus M$ is isomorphic to the n -fold direct sum $N \oplus \cdots \oplus N$ implies that $M \cong N$. \square

We introduce a quick definition of a certain class of monoids which are suitable for being the monoids of grading for graded objects, e.g. \mathbb{N} and \mathbb{Z} .

Definition 4.2.2. *We will say that a commutative monoid \mathbb{M} is finitely-generated and weakly free, or FGWF for short, if \mathbb{M} is isomorphic to a finite Cartesian product of copies of \mathbb{N} and \mathbb{Z} .*

Definition 4.2.3.

- **(Ungraded case.)** Suppose $B \rightarrow C$ is a surjective morphism of rings. Let \mathcal{B} be the category of finitely generated left B -modules and module morphisms, and \mathcal{C} the category of finitely generated left C -modules and module morphisms. We say that B is functorially relatively quasi-Frobenius over C if the abelian category \mathcal{B} is functorially relatively quasi-Frobenius over the abelian category \mathcal{C} .
- **(Graded case.)** Suppose \mathbb{M} is an FGWF monoid and suppose $B \rightarrow C$ is a surjective grading-preserving morphism of \mathbb{M} -graded rings. Let \mathcal{B} be the category of finitely generated \mathbb{M} -graded left B -modules and grading-preserving module morphisms, and \mathcal{C} the category of finitely generated \mathbb{M} -graded left C -modules and grading-preserving module morphisms. We say that B is \mathbb{M} -graded functorially relatively quasi-Frobenius over C if the abelian category \mathcal{B} is functorially relatively quasi-Frobenius over the abelian category \mathcal{C} .

Theorem 4.2.4. Let k be a finite field and let

$$A \rightarrow B$$

be a map of k -algebras which are finite-dimensional as k -vector spaces. Suppose A is augmented. We write C for the algebra $C \cong B \otimes_A k$. Suppose the following conditions are satisfied:

- B is free as a right A -module,
- the kernel of the projection $B \rightarrow C$ is contained in the Jacobson radical of B ,
- B and C are quasi-Frobenius rings,
- every projective right C -module is free, and
- B is relatively quasi-Frobenius over C .

Then the induction (base-change) maps from A to B and from B to C on the categories of finitely-generated modules over each of these Hopf algebras induce maps of stable G -theory spaces, and we have a homotopy fibre sequence:

$$|wS \cdot \mathcal{G}_{st}(A)| \rightarrow |wS \cdot \mathcal{G}_{st}(B)| \rightarrow |wS \cdot \mathcal{G}_{st}(C)|,$$

hence a long exact sequence in stable G -theory groups

$$\cdots \rightarrow (G_{st})_{n+1}(C) \rightarrow (G_{st})_n(A) \rightarrow (G_{st})_n(B) \rightarrow (G_{st})_n(C) \rightarrow (G_{st})_{n-1}(A) \rightarrow \dots$$

Proof. Thm. 3.3.5 gives us that the homotopy fiber of the map

$$|wS \cdot \mathcal{G}_{st}(B)| \rightarrow |wS \cdot \mathcal{G}_{st}(C)|$$

is $|wS \cdot \mathcal{X}|$, where \mathcal{X} is the Waldhausen category of finitely generated B -modules M such that $M \otimes_B C$ is a projective C -module, with cofibrations the injections and weak equivalences the stable equivalences. Now the results of [12] show that, with the assumptions we have made, a B -module M is in the image of the base change functor from finitely generated A -modules to finitely-generated B -modules if and only if $M \otimes_B C$ is a free right C -module. Since we have assumed that all projective C -modules are free, this means the base-change functor from finitely-generated right A -modules to finitely-generated right B -modules lands in \mathcal{X} . Furthermore, Lemma 4.2.1 implies that this functor is an embedding of categories, hence an equivalence of categories. Since B is assumed free over A , it is flat, so base-change preserves and reflects injections, i.e., cofibrations; and the projective A -modules are precisely those which become projective B -modules under base change, so base-change from A to B preserves and reflects stable equivalences. Hence \mathcal{X} is equivalent, as a Waldhausen category, to the stated Waldhausen category of finitely generated A -modules. \square

A special case of Thm. 4.2.4 is the case of $A \rightarrow B \rightarrow C$ being an extension of co-commutative connected Hopf algebras over a finite field, in which all of the conditions of Thm. 4.2.4 are automatically satisfied except for the relatively quasi-Frobenius condition. See [13] for definitions and basic properties of connected Hopf algebras extensions, and [8] for proofs that these conditions hold in this case.

Corollary 4.2.5. *Let k be a finite field and $A \rightarrow B \rightarrow C$ an extension of co-commutative connected Hopf algebras over k which are finite-dimensional as k -vector spaces. Suppose B is relatively quasi-Frobenius over C . Then we get a homotopy fibre sequence*

$$(4.10) \quad |wS \cdot \mathcal{G}_{st}(A)| \rightarrow |wS \cdot \mathcal{G}_{st}(B)| \rightarrow |wS \cdot \mathcal{G}_{st}(C)|,$$

and a long exact sequence in stable G -theory groups:

$$(4.11) \quad \cdots \rightarrow (G_{st})_{n+1}(C) \rightarrow (G_{st})_n(A) \rightarrow (G_{st})_n(B) \rightarrow (G_{st})_n(C) \rightarrow (G_{st})_{n-1}(A) \rightarrow \cdots$$

Remark 4.2.6. Cor. 4.2.5 makes it clear that the issue of when B is relatively quasi-Frobenius over C is an important one. We are currently preparing a paper which includes a proof that, when $A \rightarrow B \rightarrow C$ is an extension of co-commutative connected finite-dimensional Hopf algebras over a finite field k , as long as A is isomorphic to $k[x]/x^2$, then B is relatively quasi-Frobenius over C . We also have a proof that this is true when A is isomorphic to $k[x, y]/(x^2, y^2)$. There is evidence that the deformation-theoretic method we use to prove that the relative quasi-Frobenius conditions holds in these two cases can be extended to *any* connective A . If this is indeed the case, then the conclusion of Thm. 4.2.4 holds for *all* such extensions.

Thm. 4.2.4 does not necessarily hold if we assume that B is not free as a right A -module but only faithfully flat, or if we do not make enough assumptions (e.g. the finite-dimensionality assumption we have made in the statement of the theorem) to force the finitely generated A -modules to satisfy a Krull-Schmidt property. However, the failure of the theorem to hold in greater generality is very interesting: the base-change functor from finitely-generated right A -modules to finitely-generated right B -modules is not necessarily an embedding of categories in the non-free or non-Krull-Schmidt case, but one can often use the theory of twisted forms (e.g. as in [15]) to identify the fiber of this map of categories over any object in terms of an H^1 cohomology group. It would be very interesting to figure out some general result relating, on the one hand, the cohomology groups classifying the twisted forms, and on the other hand, the relevant stable G -theory groups.

As a corollary of Thm. 4.2.4 and as a special case of Cor. 3.3.8, we have—provided the necessary relatively quasi-Frobenius conditions are met, e.g. using a deformation-theoretic argument we mentioned above—a spectral sequence which computes the stable G -theory of a connected co-commutative Hopf algebra B over a finite field. The spectral sequence’s input consists of the stable G -theory of each “composition factor” Hopf algebra in a “composition series” for B .

Corollary 4.2.7. *Let k be a finite field and let B be a co-commutative connected Hopf algebra over k . Suppose n is a positive integer and we have a commutative diagram of*

co-commutative connected Hopf algebras over k

$$\begin{array}{ccc}
 A_n & \longrightarrow & B_n = B \\
 & & \downarrow \\
 A_{n-1} & \longrightarrow & B_{n-1} \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 A_1 & \longrightarrow & B_1 \\
 & & \downarrow \\
 A_0 & \xrightarrow{\cong} & B_0,
 \end{array}$$

in which each composable pair of maps

$$A_i \rightarrow B_i \rightarrow B_{i-1}$$

for $i > 0$ is an extension of connected Hopf algebras. Suppose each B_i is relatively quasi-Frobenius over B_{i-1} . Then there exists a strongly convergent spectral sequence with E_1 -term

$$E_1^{p,q} \cong (G_{st})_p(A_q) \Rightarrow (G_{st})_p(B)$$

with differentials

$$d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p-1, q+r}$$

converging to $(G_{st})_*(B)$.

Proof. This is just the usual spectral sequence of the tower of fibrations

$$\begin{array}{ccc}
 \Omega |wS \cdot \mathcal{G}(A_n)| & \longrightarrow & \Omega |wS \cdot \mathcal{G}(B_n)| \\
 & & \downarrow \\
 \Omega |wS \cdot \mathcal{G}(A_{n-1})| & \longrightarrow & \Omega |wS \cdot \mathcal{G}(B_{n-1})| \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 \Omega |wS \cdot \mathcal{G}(A_1)| & \longrightarrow & \Omega |wS \cdot \mathcal{G}(B_1)| \\
 & & \downarrow \\
 & & \Omega |wS \cdot \mathcal{G}(B_0)|
 \end{array}$$

which one knows are indeed homotopy fibre sequences from Thm 4.2.4. \square

Now we offer the graded generalization of Thm. 4.2.4. When \mathbb{M} is an FGWF monoid and A is an \mathbb{M} -graded ring, we will write $\mathcal{G}_{st}^{\mathbb{M}-gr}(A)$ for the stable G -theory Waldhausen category of the abelian category of finitely-generated \mathbb{M} -graded A -modules, and we will write $(G_{st}^{\mathbb{M}-gr})_n(A)$ for its n th homotopy group $\pi_n \Omega |wS \cdot \mathcal{G}_{st}^{\mathbb{M}-gr}(A)|$.

Theorem 4.2.8. *Suppose \mathbb{M} is an FGWF monoid (e.g. $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{Z}$). Suppose k is a finite field, A, B are \mathbb{M} -graded algebras which are finite-dimensional as k -vector spaces, and*

$$A \rightarrow B$$

is an \mathbb{M} -graded k -algebra map. Suppose A is augmented. We write C for the \mathbb{M} -graded algebra $C \cong B \otimes_A k$. Suppose the following conditions are satisfied:

- *B is free as a right A -module,*
- *the kernel of the projection $B \rightarrow C$ is contained in the Jacobson radical of B ,*
- *B and C are functorially \mathbb{M} -graded-quasi-Frobenius rings, i.e., $\mathcal{G}_{st}^{\mathbb{M}-gr}(A)$ and $\mathcal{G}_{st}^{\mathbb{M}-gr}(B)$ are each functorially quasi-Frobenius,*
- *every projective right \mathbb{M} -graded C -module is free, and*
- *B is functorially relatively \mathbb{M} -graded-quasi-Frobenius over C .*

Then we get the homotopy fiber sequence

$$|wS \cdot \mathcal{G}_{st}^{\mathbb{M}-gr}(A)| \rightarrow |wS \cdot \mathcal{G}_{st}^{\mathbb{M}-gr}(B)| \rightarrow |wS \cdot \mathcal{G}_{st}^{\mathbb{M}-gr}(C)|,$$

hence a long exact sequence in stable G -theory groups

$$\cdots \rightarrow (\mathcal{G}_{st}^{\mathbb{M}-gr})_{n+1}(C) \rightarrow (\mathcal{G}_{st}^{\mathbb{M}-gr})_n(A) \rightarrow (\mathcal{G}_{st}^{\mathbb{M}-gr})_n(B) \rightarrow (\mathcal{G}_{st}^{\mathbb{M}-gr})_n(C) \rightarrow (\mathcal{G}_{st}^{\mathbb{M}-gr})_{n-1}(A) \rightarrow \cdots$$

Proof. Same proof as Thm. 4.2.4. □

REFERENCES

- [1] Hyman Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
- [2] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.
- [3] Carl Faith. Embedding modules in projectives: A report on a problem. In *Advances in Non-Commutative Ring Theory*, volume 951, pages 21–40. Lecture Notes in Mathematics, 1982.
- [4] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [5] T. Y. Lam. *Exercises in modules and rings*. Problem Books in Mathematics. Springer, New York, 2007.
- [6] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.
- [7] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [8] H. R. Margolis. *Spectra and the Steenrod algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category.
- [9] Daniel Quillen. On the cohomology and K -theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.
- [10] Daniel Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
- [11] Andrew Salch. Homotopy colimits in stable representation theory. *available on arXiv*.
- [12] Andrew Salch. A recognition principle for the existence of descent data. *available on arXiv*.
- [13] William M. Singer. Extension theory for connected Hopf algebras. *J. Algebra*, 21:1–16, 1972.
- [14] Friedhelm Waldhausen. Algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
- [15] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.